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LETTER TO THE EDITOR

A limiter of critical exponent universality†

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Abstract. We show that in three dimensions the consideration of single-spin distributions containing a factor of $\exp(\lambda s^6)$ can lead to statistical mechanical models of dramatically different character from those defined by the usual field theory picture of the renormalisation group theory of critical phenomena. These models plainly fall outside the usual, and previously expected, universality class.

In a recent paper (Baker and Johnson 1984) it was demonstrated that the notion, that the critical exponents in the continuous-spin Ising model are universal, is at variance with strong numerical evidence for an explicit two-dimensional example. The idea that there might be universal behaviour of fluids near their critical points is a very old one, going back as the 'law of corresponding states' to the previous century. More recently the idea has been mathematically formulated (Griffiths 1970) and used to develop a better phenomenological understanding of critical phenomena (Kadanoff 1971). The universality hypothesis is (Kadanoff 1976) 'that all critical problems may be divided into classes differentiated by: (a) the dimensionality of the system; (b) the symmetry group of the order parameter; and (c) perhaps other criteria. Within each class, the critical properties are supposed to be identical, or at worst, to be a continuous function of a very few parameters'. The point is that when the correlation length is very large only a small number of general features of the Hamiltonian matter.

Specifically for systems with a one-dimensional order parameter it was thought that critical phenomena could be understood in terms of λs^4 Euclidean, Boson, quantum field theory. This argument is given by Wilson and Kogut (1974) where it is made plain that the work is being carried out near four dimensions and in the context of a perturbation expansion of the log single-spin distribution function. This line of reasoning forms the starting point for the expansion in powers of $\varepsilon = 4-d$ (Wilson and Fisher 1972, Wilson 1972, Brézin *et al* 1976) and the expansions in powers of the field theoretic renormalised coupling constant in three dimensions (Parisi 1973, Baker *et al* 1976, 1978, Le Guillou and Zinn-Justin 1977). The interpretation of these calculations as providing, for example, the critical behaviour of the three-dimensional Ising model, depends on the universality of hypothesis as stated above without further differentiation (c).

It is well known that s^6 is a marginal quantity in three dimensions (see, for example, Ma 1976). Theories including it have been studied in connection with tricritical

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phenomena (Wegner and Riedel 1973, Stephen *et al* 1975). Pisarski (1982) has studied the large N-limit of $(s^6)_3$ field theory, where N is the number of components of the order parameter. So far as I have been able to determine, no one has fully investigated the effect of such a term on the differentiation of universality classes.

By considering ferromagnetic, continuous-spin Ising models whose single-spin distribution is of the form $\exp(as^2+bs^4+cs^6)$ we are able to demonstrate the existence of models whose truncated four-spin correlation functions are dramatically different, in fact of the opposite sign entirely, from those of models whose single-spin distribution function is restricted to be of the form $\exp(as^2+bs^4)$. These models, arguing even from the opposite sign of the four-spin correlation function alone, must fall outside the universality class produced by the usual, field theoretic picture of the renormalisation group theory of critical phenomena based on λs^4 Euclidean, Boson quantum field theory.

We introduce the lattice cut-off version of $\lambda_6{:}\phi^6{:}_d$ Euclidean, Boson field theory as given by the partition function

$$Z=M^{-1}\int_{-\infty}^{+\infty}\cdots\int \exp\bigg[-\sum_{r} a^d\bigg(\sum_{\{\delta\}}\frac{(\phi_r-\phi_r+\phi_\delta)^2}{a^2}+m_0^2{:}\phi_r^2{:}+\lambda_4{:}\phi_r^4{:}$$

$$+\lambda_6{:}\phi_r^6{:}+H_r\phi_r\bigg)\bigg]\prod_r \mathrm{d}\phi_r, \tag{1}$$

where : : denotes the normal ordered product, a is the lattice spacing, the sum over $\boldsymbol{r}$ is over the d-dimensional hyper-simple cubic lattice, the sum over $\{\boldsymbol{\delta}\}$ is over half the nearest neighbours, and M^{-1} is a formal normalisation constant. Specifically (Baker 1975)

$$\begin{aligned}&{:}\phi^2{:}=\phi^2-C, \qquad {:}\phi^4{:}=\phi^4-6C\phi^2+3C^2,\\ &{:}\phi^6{:}=\phi^6-15C\phi^4+45C^2\phi^2-15C^3,\end{aligned} \tag{2}$$

where C is the commutator,

$$C=\int_{-1/2a}^{1/2a}\cdots\int\frac{\mathrm{d}\boldsymbol{k}}{m_0^2+4a^{-2}\sum_{\{\delta\}}\sin^2(\pi\boldsymbol{k}\cdot\boldsymbol{\delta})}. \tag{3}$$

Note: $|\boldsymbol{\delta}|=a$. It is convenient to rewrite (1) as a continuous-spin, Ising model

$$Z=M^{-1}\int_{-\infty}^{+\infty}\cdots\int\prod_r \mathrm{d}s_r \exp\bigg[\sum_r\bigg(K\sum_{\{\delta\}}s_r s_{r+\delta}-\tilde{A}s_r^2-\tilde{g}_0 s_r^4-\tilde{\lambda}s_r^6+\tilde{H}_r s_r\bigg)\bigg] \tag{4}$$

with the connections

$$\begin{aligned}&\phi_r=[K/a^{d-2}]^{1/2}s_r, \qquad \lambda_0=K^3a^{6-2d}\lambda_6,\\ &\tilde{g}_0=K^2a^{4-d}(\lambda_4-15C\lambda_6), \qquad \tilde{H}_r=(Ka^{d+2})^{1/2}H_r\\ &\tilde{A}=K(2d+m_0^2a^2-6Ca^2\lambda_4+45C^2a^2\lambda_6).\end{aligned} \tag{5}$$

The formal constant M in equation (4) has also changed from that in equation (1). Since we will treat in this paper mainly quantities which are independent of the amplitude renormalisation, we impose the following one

$$1=\langle s^2\rangle_{H=K=0}=\frac{\int_{-\infty}^{+\infty}\cdots\int s^2\exp(-\tilde{A}s^2-\tilde{g}_0s^4-\tilde{\lambda}_0s^6)\,\mathrm{d}s}{\int_{-\infty}^{+\infty}\cdots\int \exp(-\tilde{A}s^2-\tilde{g}_0s^4-\tilde{\lambda}_0s^6)\,\mathrm{d}s} \tag{6}$$

which defines the function $\tilde{A}(\tilde{g}_0, \tilde{\lambda}_0)$, and absorbs the extra freedom from our introduction of the parameter K. K has the interpretation of an inverse temperature.

The first step in our investigation is to compute the renormalised coupling constant (let $\tilde{H}_r = H$ henceforth), which is defined as

$$g=-(\partial^2\chi/\partial H^2)/[\chi^2\xi^d], \tag{7}$$

where in terms of expectation values with respect to the partition function (4), χ is the magnetic susceptibility, and ξ is the correlation length in units of the lattice spacing. Thus, explicitly for K less than the critical value K_c

$$\chi=\sum_r\langle s_0s_r\rangle, \qquad \xi^2=\left[\sum_r\left(\frac{r}{a}\right)^2\langle s_0s_r\rangle\right]\Big/(2\,d\chi), \tag{8}$$

$$\frac{\partial^2\chi}{\partial H^2}=\sum_{r,t,u}(\langle s_0s_rs_ts_u\rangle-\langle s_0s_r\rangle\langle s_ts_u\rangle-\langle s_0s_t\rangle\langle s_rs_u\rangle-\langle s_0s_u\rangle\langle s_ts_r\rangle).$$

We begin this computation by a first-order perturbation expansion in $\tilde{g}_0$ and $\tilde{\lambda}_0$ small. We proceed by expanding powers of K, and retain all the terms which contain $\tilde{g}_0$ or $\tilde{\lambda}_0$ linearly. We employ the linked-cluster expansion method (Wortis 1974). This method expresses the series directly in terms of the cumulants of the single-spin distribution, equation (6). A little calculation with $H=0$ shows, after the elimination of $\tilde{A}$ by means of condition (6), that the $2n$th cumulants, M_{2n}^0 in the usual notation, are, to linear order in $\tilde{g}_0$ and $\tilde{\lambda}_0$,

$$M_2^0=1.0, \qquad M_4^0=-24\tilde{g}_0-360\tilde{\lambda}_0,$$
$$M_6^0=-720\tilde{\lambda}_0, \qquad M_{2n}^0=0,\ n\geqslant 4. \tag{9}$$

M_0^0 and all M_n with n odd vanish. We will need χ and ξ to zero order in $\tilde{g}_0$ and $\tilde{\lambda}_0$ for this calculation and $\partial^2\chi/\partial H^2$ to linear order in $\tilde{g}_0$ and $\tilde{\lambda}_0$. Thus, we need to consider only those high-temperature graphs which have any number of vertices which are the meet of two lines plus one four-line or one six-line vertex. Baker and Kincaid (1981) have computed the four-line vertex case. I remind the reader that a magnetic field derivative at a point is equivalent to a line in this counting. There are, of course, four derivatives of ln Z in $\partial^2\chi/\partial H^2$. The class of graphs with one six-line vertex to be considered here is just: (a) polygons with one root at which the four derivatives act and (b) polygons with one root at which one, two, three or four linear chains are attached. In case (b) one derivative acts at the free end of each linear chain and the remainder act at the root point. As an aside, the point does not count as a polygon but the single, double-bond does. Using the results of Baker and Kincaid (1981), that the generating function of the linear chains (free multiplicities are used here) is $[2dK/(1-2dK)]$, and the polygon generating function is

$$P_d(K)=\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int\left(\prod_{\tau=1}^{d}\mathrm{d}\theta_\tau\right)\Big/\left[1-\left(2K\sum_{\tau=1}^{d}\cos\theta_\tau\right)^2\right]-1, \tag{10}$$

we obtain, using the standard sort of combinatorical factors (Wortis 1974),

$$\frac{\partial^2\chi}{\partial H^2}=\frac{-1}{(1-2dK)^4}\{24\tilde{g}_0+360\tilde{\lambda}_0+30\tilde{\lambda}_0P_d(K)[24-168dK$$
$$+464(dK)^2-576(dK)^3+272(dK)^4]\}+\mathrm{O}(\tilde{g}_0^2,\tilde{g}_0\tilde{\lambda}_0,\tilde{\lambda}_0^2). \qquad (11)$$

This equation can be combined with (Baker and Kincaid 1981)

$$\begin{aligned}\chi&=(1-2dK)^{-1}+\mathrm{O}(\tilde{g}_0,\tilde{\lambda}_0),\\ \xi^2&=K(1-2dK)^{-1}+\mathrm{O}(\tilde{g}_0,\tilde{\lambda}_0),\\ m^2a^2\xi^2&=1,\end{aligned} \qquad (12)$$

where m is the field theory renormalised mass and it measures the decay of the two-point correlation function as proportional to $\exp(-rm)$. If we fix our unit of length by $m=1$, then equations (7), (11) and (12) give

$$g=\xi^{4-d}K^{-2}\{24\tilde{g}_0+360\tilde{\lambda}_0+30\tilde{\lambda}_0P_d(K)[24-168dK+464(dK)^2$$
$$-576(dK)^2+272(dK)^3]\}+\mathrm{O}(\tilde{g}_0,\tilde{g}_0\tilde{\lambda}_0,\tilde{\lambda}_0^2). \qquad (13)$$

We can make a few observations. First we now specialise to $d=3$. Clearly $K_c=(2d)^{-1}$. It is simple to show from (10) that $P_3(K_c-\varepsilon)=\text{constant}\ -\text{constant}\sqrt{\varepsilon}$. Since this is the case, for any $K_0<K_c$ we can pick a value of $\tilde{\lambda}_0$ of the same order of magnitude as $\tilde{g}_0$ so that $g=0$, to leading order for that particular value of K_0. We observe, because we may compute that the coefficient of $\tilde{\lambda}_0$ inside { } in equation (13) is monotonic increasing in K, that $g<0$ for $0<K<K_0$ and $g>0$ for $K_0<K<K_c$. If we choose $\tilde{g}_0=-G\xi^{-1}K^2$, with G small but of order one with respect to ξ^{-1}, and

$$\tilde{\lambda}_0/\tilde{g}_0=-4/(60+5P_3(\tfrac{1}{6})) \qquad (14)$$

then we find in leading order that $g\propto-\sqrt{K_c-K}\propto-\xi^{-1}$ as $\xi\to\infty$, as distinguished from $g=+24\lambda_4$ with the choice $\tilde{\lambda}_0\equiv0$. We point out by (5) that, as for the choice (14) $\tilde{\lambda}_0\propto\xi^{-1}$, so therefore, is λ_6 also. Thus, λ_6 vanishes in the critical point limit, although as C in three dimensions is proportional to ξ, λ_6C, which enters the definition of $\tilde{g}_0$, has a finite limit. These two models are manifestly different in that for $\tilde{\lambda}_0\equiv0$, g is positive and goes to a non-zero value in the limit as the critical point is approached (continuum limit), while for choice (14) g is negative and vanishes in the critical point limit.

I argue that, by an appropriate choice of $\tilde{g}_0(\tilde{\lambda}_0)$, the condition $g=0$ can be met for a given K_0, $0<K_0<K_c(\tilde{g}_0,\tilde{\lambda}_0)$, say $K_0=fK_c$, for all $0<\tilde{\lambda}_0\leqslant\infty$, where for example $0<f<1$ is any fixed number. Certain basic properties are known, χ and ξ^2 are positive throughout the range by Griffith's first inequality (Ginibre 1970) and here all relevant quantities are continuous functions of $\tilde{g}_0$, $\tilde{\lambda}_0$ as long as $K<K_c$ (Baker 1975). For any single-spin distribution the expansion in powers of K is given through tenth order by Baker and Kincaid (1981). For the simple cubic lattice, it begins

$$\partial^2\chi/\partial H^2=(I_4-3I_2^2)+24I_2(I_4-3I_2^2)K+3(-303I_2^4+87I_2^2I_4+I_2I_6+3I_4^2)K^2+\mathrm{O}(K^3) \qquad (15)$$

where I_{2n} is the $2n$th moment of the distribution of equation (6). From equation (15), it follows that if $I_4=3I_2^2$, there is a double-zero at $K=0$. This condition is always possible as for fixed $\tilde{\lambda}_0$, $\tilde{g}_0 \to +\infty$ forces $I_4 \to I_2^2$ and $\tilde{g}_0 \to -\infty$ forces I_4/I_2^2 to infinity, under condition (6).

For the limit $\tilde{\lambda}_0 \to \infty$, the family of single-spin distributions ($a \geqslant 1$) becomes

$$\tfrac{1}{2}(\delta(s-a)+\delta(s+a))/a^2+[(a^2-1)/a^2]\delta(s) \tag{16}$$

where $\delta(x)$ is the Dirac delta-function and the moments are $I_{2n} = a^{2(n-1)}$, $n \geqslant 1$, $I_0 = 1$. The series for the simple cubic lattice is

$$\begin{aligned}\partial^2\chi/\partial H^2 = -3 + a^2 + (-72+24a^2)K + (-909+261a^2+12a^4)K^2 \\ +(-8568+1812a^2+288a^4+4a^6)K^3+\dots\end{aligned} \tag{17}$$

Numerical analysis indicates, starting with the spin-$\frac{1}{2}$ Ising value $a=1$, that $\partial^2\chi/\partial H^2$ is negative at $K=0$ and that there are two complex zeros near the imaginary K axis which collide at $K=0$ when $a=\sqrt{3}$ and move out of the positive and negative K axes with increasing a. By the time $a \simeq 5$ all the known terms of (17) are positive and there is clear numerical evidence that $\partial^2\chi/\partial H^2$ is diverging to plus infinity rather than minus infinity as for $a=1$. Hence, by continuity arguments there exists a least upper bound a_0 for all a's which correspond to $\partial^2\chi/\partial H^2=0$ for $K_0 < K_c(\tilde{\lambda}_0=\infty)$. Thus, for that a_0 we have at $\tilde{\lambda}_0=\infty$, a model of the same character, i.e., $g<0$ for $K<K_c$ and $g\to 0$ as $K \to K_c$ as for choice (14) for $\tilde{\lambda}_0 \simeq 0$. This model is a completely legitimate, ferromagnetic Ising model (something like a spin-one model) and is in sharp contrast to the spin-$\frac{1}{2}$ Ising model for which $g>0$ for $K<K_c$.

We thus have seen that Kadanoff's (1976) '(c) perhaps other criteria' is not empty for the family of continuous-spin Ising model and must take account at least of s^6 effects.

To return to results of Baker and Johnson (1984) that γ for the two-dimensional s^4 'border', continuous-spin Ising model differs from that for the spin-$\frac{1}{2}$ Ising model, I remark that the extra freedom allowed in the model herein described may well apply to a pure s^4 model as well because, following renormalisation group ideas, even if s^6 terms are not present in the original single-spin distribution function, they will surely be generated by the successive elimination of the short-range degrees of freedom in the system and hence may account for the limitation of universality they observed. The analysis of their two-dimensional system is more complex than the three-dimensional case treated here because all the simple powers s^{2n} are relevant, but it seems implausible to suppose that universality is less affected there than in three dimensions.

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